Quasi-isometrically embedded subgroups of braid and diffeomorphism groups

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Abstract We show that a large class of right-angled Artin groups (in particular, those with planar complementary defining graph) can be embedded quasi-isometrically in pure braid groups and in the group $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ of area preserving diffeomorphisms of the disk fixing the boundary (with respect to the L^2-norm metric); this extends results of Benaim and Gambaudo who gave quasi-isometric embeddings of F_n and $\mathbb{Z}^n$ for all $n > 0$. As a consequence we are also able to embed a variety of Gromov hyperbolic groups quasi-isometrically in pure braid groups and in the group $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$. Examples include hyperbolic surface groups, some HNN-extensions of these along cyclic subgroups and the fundamental group of a certain closed hyperbolic 3-manifold.

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1 Introduction

We recall that a right-angled Artin group is a group which can be described by a presentation with a finite number of generators, and a finite list of relations, each of which states that some pair of generators commutes. Thus free groups and free abelian groups are examples of right-angled Artin groups. To any simplicial graph Δ we can associate a right-angled Artin group $G(\Delta)$ by having generators corresponding to the vertices of Δ, and a commutation relation between two generators if and only if the corresponding vertices of Γ are connected by an edge. Thus, if Δ has vertex set $\{1, 2, .., n\}$ then

$$G(\Delta) = \langle a_1, a_2, .., a_n \mid a_i a_j = a_j a_i \text{ for each edge } \{i, j\} \text{ of } \Delta \rangle \, .$$

Let Γ, Γ' denote groups which are equipped with left-invariant metrics d, d' respectively. By a *quasi-isometric embedding* of (Γ, d) into (Γ', d') we shall mean a group homomorphism $\phi\colon \Gamma \to \Gamma'$ which is injective and which, for some uniform constants $\lambda > 1$ and $C > 0$, satisfies the inequality

$$\frac{1}{\lambda}d(g,h) - C \leq d'(\phi(g), \phi(h)) \leq \lambda d(g,h) + C\,, \qquad \text{for all } g, h \in \Gamma\,.$$

Throughout the paper, when a finite generating set is given for a group, the group shall always be equipped with the word metric, even when no metric is specified. We note that the word metric is quasi-isometrically invariant under different choices of finite generating set.

The aim of this paper is to prove that each member of a large class of right-angled Artin groups, which we shall call *planar type* right-angled Artin groups, embeds *quasi-isometrically* in some pure braid group PB_m, as well as in the group $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ of area-preserving diffeomorphisms of the unit disk, equipped with the so-called "hydrodynamical" or L^2-norm metric. Quite independently one may observe that many interesting groups may be embedded quasi-isometrically in right-angled Artin groups. As a corollary, then, we obtain that all surface groups, with the exception of the three simplest non-orientable surfaces, as well as at least one hyperbolic 3-manifold group, embed quasi-isometrically in PB_m, for some m, and in $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$.

In [4] the authors showed that each right-angled Artin group of planar type embeds in a pure braid group PB_m, for some m depending on the defining graph Δ. However, it was not clear whether this embedding is quasi-isometric. In the present paper we modify the construction in order to give quasi-isometric embeddings. Our techniques also yield quasi-isometric embeddings of arbitrary right-angled Artin groups into closed surface mapping class groups. (See Theorem 4 and Corollary 5).

The initial motivation for the present paper, however, came from the work [2] of Benaim and Gambaudo, who introduce the hydrodynamical metric d_{hydr} on the group $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ of volume preserving diffeomorphisms of the closed disk D^2 (see Section 3 for details). Observing that the metric d_{hydr} is unbounded they proposed a study of the large-scale properties of $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ with respect to this metric. Their main result in this direction states that, for any n, the free abelian and free groups, $\mathbb{Z}^n$ and F_n, embed quasi-isometrically in $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$. Adapting their techniques to the case of right-angled Artin groups we are able to show that all of the examples referred to above (planar type right-angled Artin groups, surface groups, and other hyperbolic group examples) may also be embedded quasi-isometrically in $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ with respect to d_{hydr}.

It should also be stressed that we do not actually construct a quasi-isometric embedding of PB_m in $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$. In fact, it is unknown whether PB_m can be embedded as a subgroup of $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ or $\mathit{Homeo}(D^2, \partial D^2, \mathit{vol})$, and it is rather unlikely that this is possible in any natural way. For instance, it follows from the work of Morita [10] that there is no group-theoretic section of the natural homomorphism $\mathcal{P}_m \to PB_m$ where $\mathcal{P}_m < \mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ denotes a subgroup of the area-preserving diffeomorphisms which are fixed on a given set of m disjoint closed disks in the interior of D^2.

The plan of this paper is as follows. In section 2 we define planar right-angled Artin groups, and prove that they embed quasi-isometrically in a pure braid group PB_m for large enough m. This proof is the heart of the paper. In section 3 we deduce a quasi-isometric embedding of planar right-angled Artin groups in $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$. This section will not come as a surprise to any reader of [2]. Finally, in section 4 we build on the work in [4] in order find many interesting quasi-isometrically embedded subgroups of PB_m and $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$.

2 Planar right-angled Artin groups in pure braid groups

In this section we define the notion of "planarity" for right-angled Artin groups and show that a right-angled Artin group of planar type $G(\Delta)$ may be embedded quasi-isometrically in an m-strand pure braid group PB_m (where m depends on the defining graph Δ) – see Corollary 5 (2). More generally, our techniques show that any right-angled Artin group (not necessarily of planar type) may be embedded quasi-isometrically in the mapping class group $\mathit{Mod}(S)$ of an orientable punctured surface, surface with boundary or closed surface (with implicit restrictions on the genera in each case) – see Theorem 4 and Corollary 5 (1). In achieving these results we do not overly concern ourselves with the problem of minimising the number m of stands in the target pure braid group or the genera, or number of punctures or boundary components, of the surface S when the target is the mapping class group $\mathit{Mod}(S)$. It would nevertheless be interesting to further understand for which m the group PB_m, and for which surfaces S the group $\mathit{Mod}(S)$, admits a (quasi-isometric) embedding of a given right-angled Artin group. We first fix some notation.

Notation $(\boldsymbol{\mathit{Mod}}(S),\ \boldsymbol{\mathit{PMod}}(S))$ Let S denote a (not necessarily connected) *finitely punctured compact orientable surface*: S is homeomorphic to

$S_0 \setminus P$ where S_0 is a compact orientable surface with boundary ∂S_0, and P is a finite set of points in $S_0 \setminus \partial S_0$. We note that the boundary of S is just that of S_0, namely $\partial S = \partial S_0$.

We shall write simply $Mod(S)$ for the orientable mapping class group of S relative to the boundary. That is, the diffeomorphisms of S are supposed to fix the boundary pointwise, and they are equivalent if they are related by a diffeotopy that is the identity on the boundary. We shall also write $PMod(S)$ for the subgroup of $Mod(S)$ generated by Dehn twists. This is simply the finite index normal subgroup of $Mod(S)$ whose elements leave invariant the components of S and do not permute the elements of the puncture set P [5].

Note that if S is a disjoint union of surfaces S_1 and S_2 then we simply have $Mod(S) \cong Mod(S_1) \times Mod(S_2)$.

2.1 Planarity, circle diagrams, and the basic representation

Any finite collection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of connected compact subsets of $\mathbb{R}^2$ (or D^2 or S^2) determines a *non-incidence* graph $\Delta_{\mathcal{C}}$, defined to be the simplicial graph with vertex set $\mathcal{C}$ and edges $\{C_i, C_j\}$ whenever $C_i \cap C_j = \emptyset$.

Note that any collection of plane diffeomorphisms $\phi_1, \ldots, \phi_n$ with ϕ_i trivial outside a regular neighbourhood of the set C_i, for each $i = 1, \ldots, n$, will generate a homomorphic image of the right-angled Artin group $G(\Delta_{\mathcal{C}})$ associated to the graph $\Delta_{\mathcal{C}}$.

Definition 1 (Circle diagram; planar type.) We say that the right angled Artin group $G(\Delta)$ is of *planar type* if Δ is isomorphic to the non-incidence graph $\Delta_{\mathcal{C}}$ where $\mathcal{C}$ denotes a finite collection of smooth simple closed curves in general position in the interior of the disk D^2. We call $\mathcal{C}$ a *(planar) circle diagram* for $G(\Delta)$.

More generally, we may define a *circle diagram* for $G(\Delta)$ to be a collection $\mathcal{C}$ of simple closed curves in some orientable surface whose non-incidence graph $\Delta_{\mathcal{C}}$ is isomorphic to Δ. Note that one may easily find a circle diagram, in this larger sense, for an arbitrary right-angled Artin group (c.f. [4]).

If Δ is a simplicial graph then we define its *complementary* (or *opposite*) graph Δ^{op} to be the simplicial graph with the same vertex set as Δ and which has an edge between two vertices if and only if Δ does not. We recall from [4] that if the complementary defining graph Δ^{op} is planar then $G(\Delta)$ is of planar type.

We also recall the idea of the proof. An embedding of the graph $\Delta^{\rm op}$ in the plane $\mathbb{R}^2$, with vertex set $\mathcal{V} \subset \mathbb{R}^2$ gives rise to a collection of simple closed curves $\mathcal{C} = \{C_v : v \in \mathcal{V}\}$ in $\mathbb{R}^2$, where C_v is defined as the boundary of a regular neighbourhood of the union of v and one-half of each edge adjacent to v.

We stress that the planarity of $\Delta^{\rm op}$ is a sufficient, but by no means necessary condition for $G(\Delta)$ to be of planar type. This point is nicely illustrated by the example described in Section 4.4. Figure 6 shows a planar circle diagram for a graph (the 1-skeleton of an icosahedron) whose complementary graph is non-planar.

Definition 2 (Surface associated to a circle diagram.) Given a (smooth) circle diagram $\mathcal{C} = \{C_1, \ldots, C_n\}$ (in an arbitrary orientable surface) we define a compact surface with boundary $S_{\mathcal{C}}$ associated to $\mathcal{C}$, as follows.

Let S' denote a regular closed neighbourhood of the circle diagram. Thus S' is a union of annuli A_i (with A_i a regular neighbourhood of C_i). Moreover, each intersection point of the curves C_i and C_j gives rise to one square in the surface S', which is just one path component of $A_i \cap A_j$. The whole surface S' is a compact orientable surface with boundary.

In each annulus A_i we introduce a pair of distinguished points which do not lie in the intersection with another annulus. The two points must be on opposite sides of the curve C_i, as indicated in figure 1. We denote P the union of all these distinguished points. Finally, we define $S_{\mathcal{C}}$ to be the surface

$$S_{\mathcal{C}} := S' \setminus N(P)\,,$$

where $N(P)$ denotes a regular open neighbourhood of the finite set P.

We remark that in the preceding construction two annuli A_i and A_j are disjoint if and only if the corresponding generators a_i and a_j of the right-angled Artin group $G(\Delta_{\mathcal{C}})$ commute.

Definition 3 (Basic representation $G(\Delta_{\mathcal{C}}) \to \boldsymbol{Mod}(S_{\mathcal{C}})$.) To a circle diagram $\mathcal{C} = \{C_1, \ldots, C_n\}$ in an orientable surface we can associate a representation $G(\Delta_{\mathcal{C}}) \to \mathit{Mod}(S_{\mathcal{C}})$ (from the right-angled Artin group whose defining graph is the non-adjacency graph of $\mathcal{C}$ to the mapping class group of the surface $S_{\mathcal{C}}$) as follows. In each annulus A_i of S we draw smooth simple closed curves B_i, C_i and D_i as indicated in figure 1, and define the following diffeomorphism for each $i = 1, \ldots, n$:

$$f_i = \tau_{B_i} \circ \tau_{D_i}^2 \circ \tau_{C_i}^{-2} \circ \tau_{B_i} \in \mathit{Diff}(S_{\mathcal{C}}, \partial S_{\mathcal{C}})\,,$$

where τ_C denotes a smooth Dehn twist along a curve C. (We remark that the diffeomorphism $\tau_{D_i}^{-2} \circ \tau_{C_i}^2$ may be thought of as induced by the pure braid of the set P of marked points on S' which is given by moving the puncture enclosed by the curves C_i and D_i twice in an anticlockwise sense around the annulus.)

This clearly defines a homomorphism $f\colon G(\Delta_{\mathcal{C}}) \to \mathit{Diff}(S_{\mathcal{C}}, \partial S_{\mathcal{C}})$ by setting $f(a_i) = f_i$.

Whenever $S_{\mathcal{C}}$ is viewed as a subsurface of any other (not necessarily compact) surface $\widehat{S}$ the homomorphism f extends naturally to a homomorphism

$$\widehat{f}\colon G(\Delta_{\mathcal{C}}) \to \mathit{Diff}(\widehat{S}, \partial\widehat{S}),$$

where every element of the image acts by the identity on $\widehat{S} \setminus S$.

We shall denote $\varphi\colon G(\Delta_{\mathcal{C}}) \to \mathit{Mod}(S_{\mathcal{C}})$ and $\widehat{\varphi}\colon G(\Delta_{\mathcal{C}}) \to \mathit{Mod}(\widehat{S})$ the homomorphisms induced by f and $\widehat{f}$ respectively. Clearly $\widehat{\varphi}$ is obtained from φ by composing with the map $\mathit{Mod}(S_{\mathcal{C}}) \to \mathit{Mod}(\widehat{S})$ induced by the inclusion.

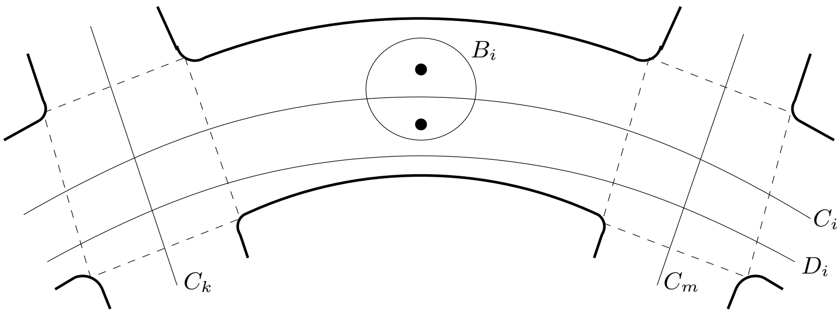


Figure 1: A segment of the ith annulus A_i, and its intersection with two other annuli, A_k and A_m. The curves B_i, C_i, D_i are indicated with solid lines, and the boundaries of the squares $A_i \cap A_k$ and $A_i \cap A_m$ with dashed lines.

We remark that in the above construction, the homeomorphism f_i, when restricted to the annulus A_i, is pseudo-Anosov. This idea, which is essential to our proof that the embedding in quasi-isometric, is inspired by [6].

Note also that throughout the whole of the above discussion we did not need to suppose that any of the surfaces are necessarily connected. If S is a disjoint union of surfaces S_1 and S_2 then we understand $\mathit{Mod}(S) \cong \mathit{Mod}(S_1) \times \mathit{Mod}(S_2)$.

2.2 Quasi-isometric embeddings

Our next aim is to prove that the basic representation $G(\Delta_{\mathcal{C}}) \to \mathit{Mod}(S_{\mathcal{C}})$ is injective and quasi-isometric. In fact, we are going to prove something slightly stronger, which is the main technical result of this section:

Theorem 4 *Let $\mathcal{C}$ be a circle diagram, $\Delta_{\mathcal{C}}$ the associated non-incidence graph, and $S_{\mathcal{C}}$ the surface of Definition 2. Suppose that $S_{\mathcal{C}}$ embeds as a subsurface of an orientable finitely punctured compact surface $\widehat{S}$ and that the embedding $S_{\mathcal{C}} \hookrightarrow \widehat{S}$ is π_1-injective on each component of $S_{\mathcal{C}}$. Then the homomorphism*

$$\widehat{\varphi}\colon\ G(\Delta_{\mathcal{C}}) \to PMod(\widehat{S})$$

is a quasi-isometric embedding of groups (in particular, an injective homomorphism).

Before passing on to the proof of this theorem, we mention some easy consequences. Firstly, since it is a finite index subgroup, the inclusion of $PMod(\widehat{S})$ into $Mod(\widehat{S})$ is a quasi-isometric embedding. It follows from the statement of Theorem 4 that we also obtain a quasi-isometric embedding of $G(\Delta_{\mathcal{C}})$ into the slightly larger group $Mod(\widehat{S})$.

Corollary 5 *Let $G = G(\Delta)$ denote a right-angled Artin group.*

(1) *The group G embeds quasi-isometrically in $Mod(S)$ for some connected closed orientable surface S (of genus depending on Δ).*

(2) *If G is of planar type then it embeds quasi-isometrically in the pure braid group PB_m (for a sufficiently large m depending on Δ).*

Proof For an arbitrary right-angled Artin group $G(\Delta)$ we may always find a circle diagram $\mathcal{C}$ on some orientable surface such that $\Delta = \Delta_{\mathcal{C}}$. Suppose that $S_{\mathcal{C}}$ has b boundary components. Then we define a π_1-injective inclusion of $S_{\mathcal{C}}$ into a closed connected surface $\widehat{S}$ by gluing $S_{\mathcal{C}}$ along its boundary to an orientable genus zero surface with b boundary components. Statement (1) now follows by applying Theorem 4.

When $G(\Delta)$ is of planar type we may find a planar circle diagram $\mathcal{C}$ with $\Delta = \Delta_{\mathcal{C}}$. The corresponding surface $S_{\mathcal{C}}$ is then also planar, and may be viewed as a subsurface of D^2. Removing a single point from each disk component of $D^2 \setminus S_{\mathcal{C}}$ yields an m-punctured closed disk $\widehat{S} \cong D^2 \setminus \{\, m \text{ points} \,\}$ (such that $\partial\widehat{S} \cong S^1$). The inclusion $S_{\mathcal{C}} \hookrightarrow \widehat{S}$ is, by construction, π_1-injective on each connected component of $S_{\mathcal{C}}$. We also recall the fact that $Mod(\widehat{S})$ is naturally isomorphic to the m-string braid group B_m, for some m, and $PMod(\widehat{S}) \cong PB_m$, the pure braid group. Statement (2) now follows from Theorem 4. □

2.3 Proof of Theorem 4

We suppose throughout that $\Delta = \Delta_{\mathcal{C}}$ where $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a smooth circle diagram in some orientable surface, and $S = S_{\mathcal{C}}$ the compact orientable surface of Definition 2. For simplicity (and without any loss of generality) we shall view $\mathcal{C}$ as being a circle diagram in the surface S. We assume that an inclusion $S \to \widehat{S}$ is given, and that the maps $f, \widehat{f}, \varphi, \widehat{\varphi}$ are as described in the preceding definitions (Subsection 2.1).

Definition 6 (Curve diagrams and intersection numbers) By a *(smooth) curve diagram* on a surface we shall mean the diffeotopy class of the union of a collection of (not necessarily disjoint) smooth simple closed curves on the surface, no two of which are homotopy equivalent.

If D, D' are two curve diagrams then we write $|D \cap D'|$ for the minimal intersection number between representatives of the diffeotopy classes D and D' which are chosen to be in transverse position with respect to one another.

We observe that the group $Mod(S)$ acts naturally (on the left) on the set of curve diagrams.

We shall first investigate the action of $G(\Delta)$ on the surface $S = S_{\mathcal{C}}$ via the homomorphism $\varphi\colon\ G(\Delta) \to Mod(S)$. We shall let $a_1, \ldots, a_n$ denote the standard generators of $G(\Delta)$. Recall that S is the union of annuli A_i, $i = 1, \ldots, n$, with a pair of open disks removed from each annulus – see Subsection 2.1.

We define an (i,k)-*intersection square* to be one connected component of $A_i \cap A_k$, the intersection of the ith and the kth annulus.

When drawing pictures of curve diagrams we shall always arrange for the curves to be *reduced* with respect to the four boundary lines of each intersection square – this means that there are no bigons enclosed between the curves and the boundary of the intersection square. This is equivalent to asking that the number of intersections between the curves and the intersection squares be minimal in the diffeotopy class of the curve diagram.

If Q is an (i,k)-intersection square, then we say that a curve diagram D traverses the square N times *in the A_i-direction* (resp. *in the A_k-direction*) if, in a version of D which is reduced with respect to the intersection squares, the following condition is satisfied: among the connected components of $Q \cap D$ there are precisely N which connect opposite sides of the square without leaving the interior of A_i (resp. A_k). See figure 2.

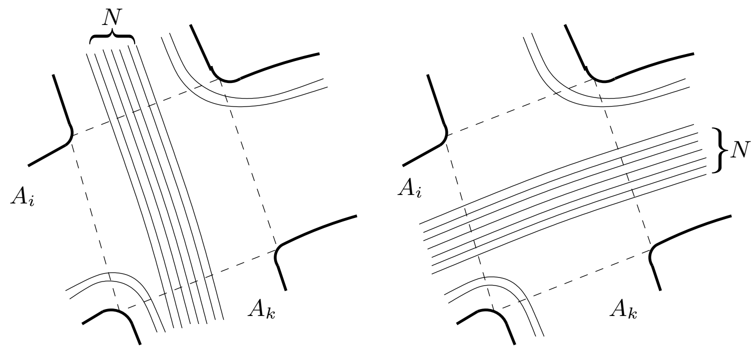


Figure 2: A curve diagram D traverses an (i,k)-square N times in the A_k-direction, and N times in the A_i direction.

Notation Let D be a curve diagram in S. For each $i = 1, \ldots, n$, write $c_i(D)$ for the largest number N such that D traverses every (i,k)-square on the annulus A_i at least N times in one or other direction. We say that D is *transverse* to A_i if there exists an (i,k)-square (where $k \in \{1, \ldots, n\}$ is such that a_i and a_k do not commute) which is traversed by D at least $c_i(D)$ times in the A_k-direction.

Lemma 7 *Let D be a curve diagram in S, a_i some generator of $G(\Delta)$, and let $D' = \varphi(a_i)^p(D)$. Suppose, furthermore, that the diagram D is transverse to the annulus A_i. Then $c_i(D') \geqslant 2^{|p|} c_i(D)$. Moreover, D' is transverse to every annulus A_k which intersects A_i (and not transverse to A_i).*

Proof Note that, since D is transverse to A_i there is some (i,k)-intersection square where D traverses at least $c_i(D)$ times in the A_k direction. We claim that the image of each of these crossing arcs under the action of $\varphi(a_i)$ will traverse each intersection square on the annulus A_i at least 2^p times in the A_i direction (once D' is placed in minimal position with respect to the boundary arcs of the intersection squares). This clearly establishes the Lemma. (To see that D' is not transverse to A_i observe that if it were then, since $D = \varphi^{-p}(a_i)(D')$, a further application of the Lemma would imply that $c_i(D) \geqslant 2^{|p|} c_i(D') \geqslant 2^{|p|} c_i(D)$, a contradiction).

Let us first consider the special case $p = 1$, that is, the action by $\varphi(a_i)$ on a single crossing arc α (the case $p = -1$ is of course similar). We first recall that $\varphi(a_i)$ is defined by the diffeomorphism

$$f_i = \tau_{B_i} \circ \tau_{D_i}^2 \circ \tau_{C_i}^{-2} \circ \tau_{B_i} \in \mathit{Diff}(S, \partial S)\,.$$

Figure 3 illustrates the different stages of the action of this element on α. At the final stage the image of α traverses each intersection square at least twice.

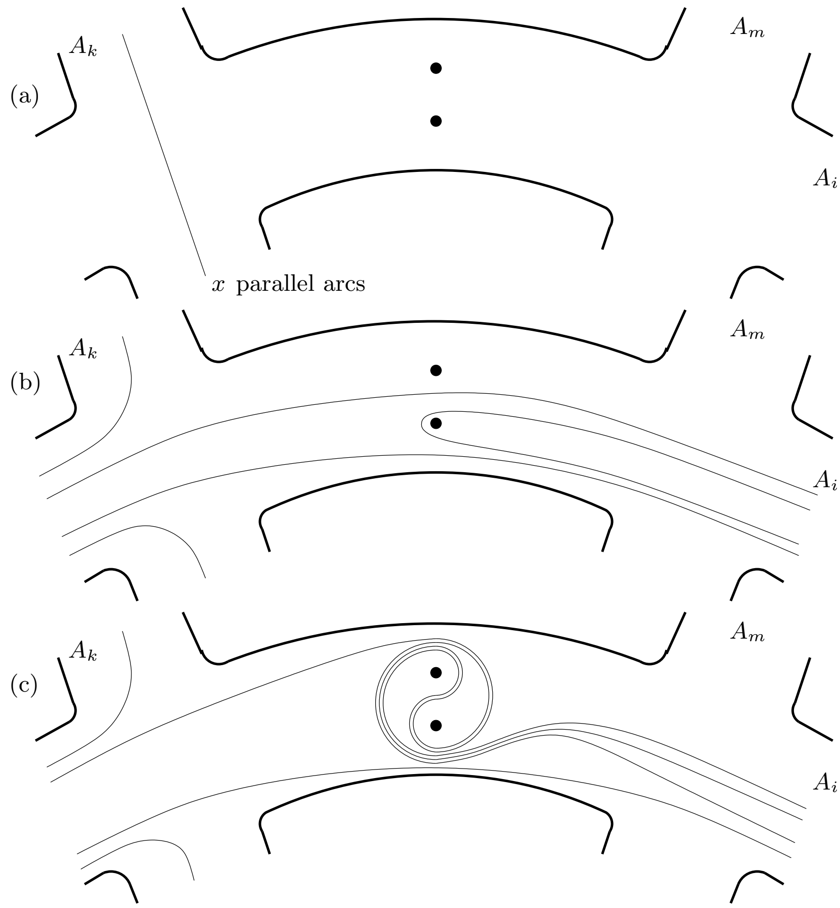


Figure 3: If there are x arcs traversing a square of $A_i \cap A_k$ in the A_k-direction, then after the action of $\widehat{f}(a_i)$ there are $2x$ arcs traversing any square of intersection involving A_i in the A_i-direction. (Part (b) shows the result of the action by $\tau_{B_i} \circ \tau_{D_i}^2 \circ \tau_{C_i}^{-2}$.)

Now we consider the more general case $p \in \mathbb{N}$ (and again, the case $-p \in \mathbb{N}$ is similar). We have just seen that the image of α under $\varphi(a_i)$ contains the segment α' shown in Figure 4(a). Now Figure 4 shows that acting once more on this curve α' by $\varphi(a_i)$ yields a curve diagram containing at least twice (in fact: five times) as many parallel copies of the same segment α' appearing as subsegments. Inductively, the curve diagram of $\varphi(a_i)^p(\alpha)$ contains at least 2^{p-1} copies of this arc α', and hence traverses each intersection square in A_i at least

2^p times in the A_i direction. □

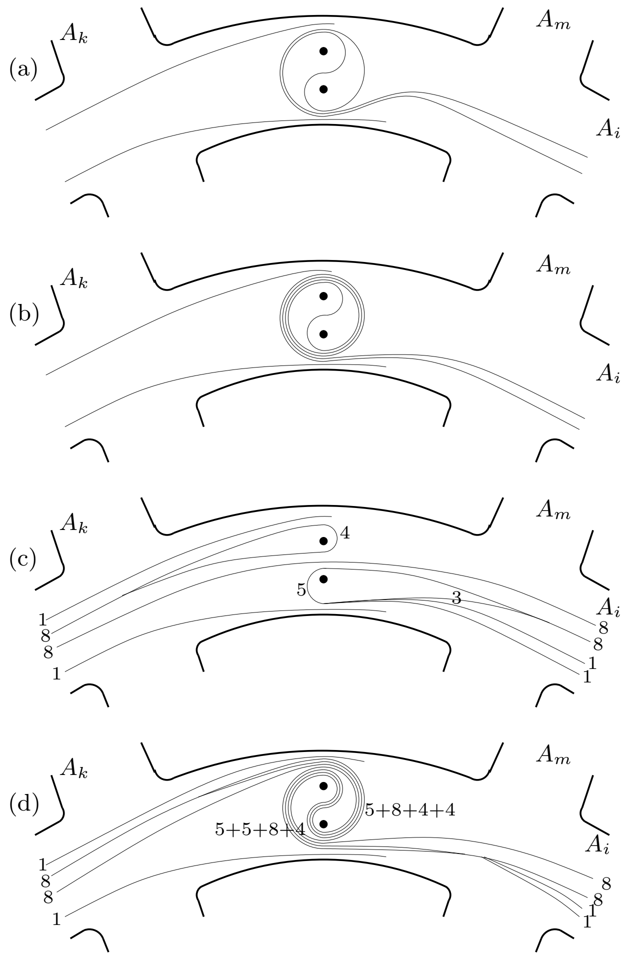


Figure 4: Acting by $\widehat{f}(a_i)$ twice in a row. (a) After the first action by $\widehat{f}(a_i)$. (b) After the action by τ_{B_i}. (c) After the action by $\tau_{C_i}^2 \tau_{D_i}^{-2}$. (d) After the action by τ_{B_i}.

We shall say that a word in the letters $a_1^{\pm 1}, \ldots, a_n^{\pm 1}$ is *reduced* if there is no shorter words in those letters representing the same element of the right-angled Artin group $G(\Delta)$. It is well-known that any two reduced words differ by finite a sequence of "shuffles": exchanges of adjacent letters $a_i^{\pm}$, $a_j^{\pm}$ where i, j span an edge in Δ (so that a_i, a_j commute). This seems to be first due to Baudisch [1]. A much more recent proof can be found in [4] (Proposition 9(i)). It follows from this result that, for each $i = 1, .., n$, the number ℓ_i of occurrences of the letters a_i and a_i^{-1} in any reduced representative is an invariant of the group element.

Lemma 8 *Let $w \in G(\Delta)$, $\phi = \varphi(w) \in Mod(S)$, and D a curve diagram in S. Suppose that D is transverse to A_i for every i for which w can be written $w = va_i^{\pm 1}$ in reduced form. Then, for all j,*

$$c_j(\phi(D)) \geqslant 2^{\ell_j(w)} c_j(D)\,,$$

where $\ell_j(w)$ denotes the number of occurrences of the letters a_j, a_j^{-1} in some, or any, reduced word representing w.

Proof Suppose $w = va_i^p$ in reduced form, where $|p|$ is as large as possible. Then if $v = ua_j$ in reduced form for some j, we have $i \neq j$. Let $D' = \varphi(a_i)^p(D)$. Since D is transverse to A_i it follows, by Lemma 7, that $c_i(D') \geqslant 2^{|p|} c_i(D)$ and, moreover, that D' is transverse to every annulus A_k which intersects A_i.

Recall that $\varphi(a_i)$ is represented by a diffeomorphism f_i which is the identity outside A_i. It follows that applying f_i to any diagram does not increase the number of crossings (in either direction) at any (j,k)-square for $j \neq i$ (and any k not adjacent to j). Since f_i^{-1} cannot increase these numbers either (for the same reason), we conclude that the diffeomorphisms f_i, f_i^{-1} can neither increase nor decrease the number of such crossings. Thus $c_j(D') = c_j(D)$ for $j \neq i$. Moreover, if D is transverse to A_j and A_j is disjoint from A_i then D' is also transverse to A_j.

We now claim that the transversality hypothesis (on D and w) applies once again to the element $v \in G(\Delta)$ and the diagram D'. For, if $v = ua_j$ in reduced form, then $j \neq i$ and either

(1) i, j are non-adjacent in Δ: here A_j intersects A_i and so D' is transverse to A_j, by the above application of Lemma 7; or

(2) i, j span an edge of Δ: here A_j and A_i are disjoint. In this case we have that $w = ua_ja_i = ua_ia_j$ in reduced form, so that D is transverse to A_j (by hypothesis). As observed above, in this case D' remains also transverse to A_j.

The Lemma now follows by applying the statement inductively to the pair v, D'. □

Recall that the curves of the original circle diagram $\mathcal{C} = \{C_1, \ldots, C_n\}$ appear as simple closed (but not disjoint) curves in S, and since no two are homotopic, their union $E = \bigcup \mathcal{C}$ is a curve diagram in S – indeed, we can think of E as the trivial curve diagram in S. We note that E is transverse to every annulus, and that $c_j(E) = 1$ for each $j = 1, \ldots, n$. By Lemma 8, this latter fact characterises E amongst all its translates by elements of the group $G(\Delta)$ (acting via φ). As a consequence, $\varphi\colon G(\Delta) \to Mod(S)$ is an injective homomorphism.

We now return to studying the homomorphism $\widehat{\varphi}\colon G(\Delta) \to PMod(\widehat{S})$. Our first observation is the following:

Proposition 9 *The homomorphism $\widehat{\varphi}\colon G(\Delta) \to PMod(\widehat{S})$ is faithful.*

Proof Since the inclusion $S \hookrightarrow \widehat{S}$ is π_1-injective on each component, the numbers $c_i(D)$ are invariants of the homotopy (or isotopy) class of the diagram D in $\widehat{S}$, just as in S. Now let $D = \widehat{\varphi}(a)(E)$. By Lemma 8, and the fact that E is transverse to every annulus, $c_j(D) \geqslant 2$ for some j unless $a = 1$. This proves injectivity of the homomorphism $\widehat{\varphi}$. □

We note that injectivity of $\widehat{\varphi}$ is also a consequence of the statement that we are about to prove, namely that $\widehat{\varphi}$ satisfies a quasi-isometric inequality. This is simply because $G(\Delta)$ is a torsion free group while, in general, any quasi-isometric homomorphism must have finite kernel.

Let $\widehat{\mathcal{C}}$ denote a finite collection of essential simple closed curves in $\widehat{S}$ which include the set $\mathcal{C} = \{C_i : i = 1, \ldots, n\}$ and such that the Dehn twists along these curves generate $PMod(\widehat{S})$. Observe that since the inclusion $S \to \widehat{S}$ is π_1-injective, no two curves of $\mathcal{C}$ are homotopic in $\widehat{S}$ (note also that no curve C_i can ever be parallel to a boundary component of S). We may therefore choose $\widehat{\mathcal{C}}$ to be minimal in the sense that no two curves are homotopic. It follows that their union $\widehat{E} = \bigcup \widehat{\mathcal{C}}$ is a curve diagram in $\widehat{S}$ (c.f. Definition 6). We note that $E \subset \widehat{E}$ is a subdiagram. We can think of $\widehat{E}$ as the extended trivial curve diagram in $\widehat{S}$.

We shall henceforth also fix the set of Dehn twists $\{\tau_C\colon C \in \widehat{\mathcal{C}}\}$ as our choice of generating set for $PMod(\widehat{S})$, and we shall write d for the word metric in $PMod(\widehat{S})$ with respect to these generators.

Definition 10 (Complexity of an element of $PMod(\widehat{S})$) Suppose that the generating curves $\widehat{\mathcal{C}}$ are chosen as above, so that $\widehat{E}$ is a curve diagram in $\widehat{S}$. If ϕ is an element of the group $PMod(\widehat{S})$ then $\phi(\widehat{E})$ denotes the curve diagram which is represented by the image of $\widehat{E}$ under any diffeomorphism representing ϕ. We define the *complexity* of a curve diagram D in $\widehat{S}$ to be

$$\mathit{complexity}(D) = \log_2(|D \cap \widehat{E}|) - \log_2(|\widehat{E} \cap \widehat{E}|)\,.$$

The complexity of an element $\phi \in \mathit{Mod}(\widehat{S})$ is defined by

$$\mathit{complexity}(\phi) = \mathit{complexity}(\phi(\widehat{E}))\,.$$

Note that the definition is normalized so that $\mathit{complexity}(\widehat{E}) = 0$.

Proof of Theorem 4 Theorem 4 is now a consequence of the faithfulness of $\widehat{\varphi}$ established in Proposition 9, and the following two propositions. Together, Propositions 11 and 12 imply that for an element $a \in G(\Delta)$ of wordlength $\ell(a)$ we have

$$\ell(a) \leqslant K_1 \cdot \mathit{complexity}(\widehat{\varphi}(a)) + K_0 \leqslant K_1 K_2 \cdot d(\widehat{\varphi}(a), 1) + K_0,$$

from which it follows that the homomorphism $\widehat{\varphi}$ is a quasi-isometric embedding.

Proposition 11 *Let $\ell(a)$ denote the length of a shortest representative word of an element $a \in G(\Delta)$. Then the complexity of $\widehat{\varphi}(a)$ grows at least linearly with $\ell(a)$. In other words*

$$\ell(a) \leqslant K_1 \cdot \mathit{complexity}(\widehat{\varphi}(a)) + K_0\,.$$

where K_0 and K_1 are some positive constants (e.g. K_1 equal to the number of generators in $G(\Delta)$, and $K_0 = K_1 \cdot \log_2(|\widehat{E} \cap \widehat{E}|)$ suffice).

Proof Recall that the subdiagram $E \subset \widehat{E}$ is a diagram in S which is transverse to every annulus, and that $c_j(E) = 1$ for each j. Let $D = \varphi(a)(E)$. Then it follows, by Lemma 8, that $c_j(D) \geqslant 2^{\ell_j(a)}$, for each $j = 1, \ldots, n$.

Suppose that C, C' are essential simple closed curves in S. Then we observe that, since the inclusion $S \to \widehat{S}$ is π_1-injective, the intersection number $|C \cap C'|$ will be the same whether measured in S or in $\widehat{S}$. (This is because, if C and C' cobound a bigon in $\widehat{S}$ then they must also cobound a bigon in S.) It follows by this reasoning that, for each $j = 1, \ldots, n$,

$$c_j(D) \leqslant |D \cap E| \leqslant |\widehat{D} \cap \widehat{E}|\,,$$

where $\widehat{D} := \widehat{\varphi}(a)(\widehat{E})$ and $D := \widehat{\varphi}(a)(E)$. Thus

$$2^{\ell(a)/n} \leqslant \max\{2^{\ell_j(a)} : j = 1, \ldots, n\} \leqslant \max\{c_j(D) : j = 1, \ldots, n\} \leqslant |\widehat{D} \cap \widehat{E}|\,,$$

and the proposition follows with $K_1 = n$ and $K_0 = K_1 \cdot \log_2(|\widehat{E} \cap \widehat{E}|)$. $\square$

Proposition 12 *The complexity of the curve diagram of an element ϕ of $PMod(\widehat{S})$ grows at most linearly with the distance of ϕ from the neutral element in the Cayley graph of $PMod(\widehat{S})$ – that is, we have*

$$\textit{complexity}(\phi) \leqslant K_2 \cdot d(\phi, 1),$$

where K_2 is a positive constant (equal to the base 2 logarithm of the number of curves in $\widehat{\mathcal{C}}$).

Proof We take the Dehn twists along the curves in $\widehat{\mathcal{C}}$ as our finite generating set for $PMod(\widehat{S})$. If τ_C is a given generator (the Dehn twist along the curve $C \in \widehat{\mathcal{C}}$) and $\widehat{D}$ a curve diagram of known complexity, then we can easily estimate the complexity of $\tau_C(\widehat{D})$. Namely, for each point of intersection of $\widehat{D}$ with C, application of τ_C may introduce at most r new points of intersection with curves in $\widehat{E}$, where

$$r = \#\{\text{ curves of } \widehat{\mathcal{C}} \text{ which intersect } C\,\} \leqslant N - 1\,,$$

where $N = |\widehat{\mathcal{C}}|$. Since $|C \cap \widehat{D}| \leqslant |\widehat{E} \cap \widehat{D}|$, we then have

$$|\widehat{E} \cap \tau_C(\widehat{D})| \leqslant |\widehat{E} \cap \widehat{D}| + (N-1) \cdot |C \cap \widehat{D}| \leqslant N \cdot |\widehat{E} \cap \widehat{D}|\,.$$

Thus $\textit{complexity}(\tau_C(\widehat{D})) \leqslant \textit{complexity}(\widehat{D}) + \log_2(N)$. By a straightforward induction on the wordlength of ϕ, we then obtain

$$\textit{complexity}(\phi) \leqslant K_2 \cdot d(\phi, 1)\,,$$

where $K_2 = \log_2(N)$. □

3 From pure braid groups to $\textit{Diff}(D^2, \partial D^2, \textit{vol})$

In the previous section we proved that there exists a homomorphic quasi-isometric embedding of any right-angled Artin group of planar type G into the pure braid group of a certain number of points $P_1, \ldots, P_m$ in a disk D^2. The number of points needed depends, of course, on the group's circle diagram. We shall assume that the disk is the disk with radius 1 and centre $(0, 0)$ in the plane, and that the distinguished points are $P_i = (\frac{i}{m-1} - \frac{1}{2}, 0)$ (with $i = 0, \ldots, m-1$).

We also recall that the homomorphism constructed in the last section factors through a certain subgroup of $\textit{Diff}(D^2, \partial D^2)$, namely the group of diffeomorphisms of the disk which fix pointwise the distinguished points.

In the current section we shall point out that there is, in fact an embedding in the group $\mathcal{P}_m$ of *volume-preserving* diffeomorphisms of the disk which, moreover, fix pointwise not only the distinguished points and the boundary of the disk, but even disks of radius r centered on each of the distinguished points, as well as a regular neighbourhood of the boundary; here r is a sufficiently small positive real number. This yields an embedding of G in the group $\mathcal{P}_m$, which is itself a subgroup of the group $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ of volume-preserving diffeomorphisms of the disk D^2 which are the identity on a neighbourhood of ∂D^2.

The aim of the current section is to prove that, if we equip $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ with the *hydrodynamical metric*, then this homomorphism $G \to \mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ is itself quasi-isometric. (This is stronger than saying that the composition $G \to \mathcal{P}_m \to PB(D^2, P_1 \cup \ldots \cup P_m)$ is quasi-isometric.)

We recall from [2] the definition of the hydrodynamical metric: if $\{\phi_t\}_{t\in[0,1]}$ is a path in $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$, then the length of this path is

$$\int_0^1 \sqrt{\int_{D^2} \left\| \frac{d\phi_t}{dt}(x) \right\|^2 dx}\ dt,$$

where the symbol $\|.\|$ denotes the Euclidean norm of a tangent vector to the disk. The hydrodynamical length $l_{\mathrm{hydr}}(\phi)$ of an element ϕ of $\mathit{Diff}(D^2, \partial D^2, \mathit{vol})$ is then the infimum length of a path from the identity map to ϕ. This defines a left-invariant metric d_{hydr} by setting $d_{\mathrm{hydr}}(\phi, \psi) = l_{\mathrm{hydr}}(\phi^{-1}\psi)$.

Now we recall a technical result of Benaim and Gambaudo (Lemma 4 in [2]). There exists a constant $K > 0$ and a function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to 0} C(r) = 0$ with the following property. Suppose that ϕ is an element of $\mathcal{P}_m$ (fixing disks of radius r around each of the distinguished points), and which represents an element of the pure braid group whose shortest expression as a product of Artin's standard generators $\sigma_1^{\pm 1}, \ldots, \sigma_{m-1}^{\pm 1}$ has length l_{Artin}, then

$$l_{\mathrm{hydr}}(\phi) \geqslant \frac{1}{K} \cdot l_{\mathrm{Artin}} \cdot (1 - C(r)) \cdot (\text{area } D_0)^2.$$

Here D_0 denotes a disk of radius r.

This technical result implies immediately the following theorem:

Theorem 13 (Benaim-Gambaudo [2]) *Suppose that f is a homomorphism from a group G to $\mathcal{P}_m$ for some choice of m fixed disks around the puncture points, and suppose that the induced homomorphism $\varphi\colon G \to PB_m$ is a quasi-isometric embedding, then so is f.*

We are now ready to prove one of the main results of this paper.

Theorem 14 *If G is a planar right-angled Artin group, then there exists a quasi-isometric embedding of G into $\mathit{Diff}(D^2, \partial D^2, vol)$.*

Proof Corollary 5(ii) gives a homomorphism $\widehat{\varphi}\colon G(\Delta) \to PB_m$ which, by the construction given in Section 2.1, factors through a homomorphism $\widehat{f}\colon G(\Delta) \to \mathit{Diff}(S, \partial S)$ where S is a compact subsurface of the m-punctured disk. We may suppose, in fact, that S is just the closed disk with a m open disks removed, and is contained in the exterior of a suitably chosen collection of m open disks of some constant radius $r > 0$. We claim that the diffeomorphisms $\widehat{f}(a_i)$ which generate the image of $\widehat{f}$ may be chosen to be area preserving. It then follows that the image of $\widehat{f}$ lies in the subgroup $\mathcal{P}_m < \mathit{Diff}(D^2, \partial D^2, vol)$ (defined by the collection of disks of radius r just mentioned). Since the induced map $\widehat{\varphi}\colon G(\Delta) \to PB_m$ is shown to be a quasi-isometric embedding (Corollary 5(ii)), the result now follows by Theorem 13.

To justify the claim, it suffices to observe that any Dehn twist about a smooth curve C in D^2 may be realised by a volume preserving diffeomorphism with support in an arbitrarily small neighbourhood of the curve C. This follows from the following two observations:

(i) a smooth embedding $c\colon S^1 \to D^2$ can be extended, for a sufficiently small $\epsilon > 0$, to a smooth area preserving embedding $S^1 \times [-\epsilon, \epsilon] \to D^2$.

(ii) if we choose a smooth function $h\colon [-\epsilon, \epsilon] \to [0, 2\pi]$ such that $h(x) = 0$ for $x < \frac{-\epsilon}{2}$, and $h(x) = 2\pi$ for $x > \frac{\epsilon}{2}$, then

$$T_h\colon S^1 \times [-\epsilon, \epsilon] \to S^1 \times [-\epsilon, \epsilon] \quad \text{such that} \quad T_h(t, s) = (t + h(s), s)$$

defines a smooth area preserving diffeomorphism which is the identity outside $S^1 \times [\frac{-\epsilon}{2}, \frac{\epsilon}{2}]$. □

4 Quasi-isometrically embedded hyperbolic subgroups

In this section we consider examples of quasi-isometric embeddings of groups in right-angled Artin groups of planar type.

4.1 Hyperbolic surface groups.

As proved in [4] every closed hyperbolic surface group with $\chi(S) \leqslant -2$ embeds quasi-isometrically in some right-angled Artin group of planar type. Therefore each of these surface groups embeds quasi-isometrically both in PB_m for some m, and in $\mathit{Diff}(D^2, \partial D^2, vol)$.

The case of an orientable surface F of genus 2 is illustrated in Figure 5. The figure describes a homomorphism $\pi_1(F) \to G(C_5)$ where C_5 denotes the 5-cycle graph with generators a, b, c, d, e. The homomorphism is determined by fixing a basepoint $\star$ and, for each loop γ based at $\star$ in F, reading the sequence of crossings with sign that γ makes with the transversely labelled curves on the surface F. We remark that this homomorphism actually projects to an embedding into the corresponding right-angled Coxeter group $W(C_5)$. (Note that $W(C_5)$ acts by isometries of the hyperbolic plane and that the surface subgroup obtained here is finite index in the Coxeter group).

Embeddings of arbritary higher genus orientable surface groups may be obtained by simply restricting to finite index subgroups of $\pi_1(F)$. The treatment of the non-orientable case is similar but more complicated to describe (see [4], Section 4).

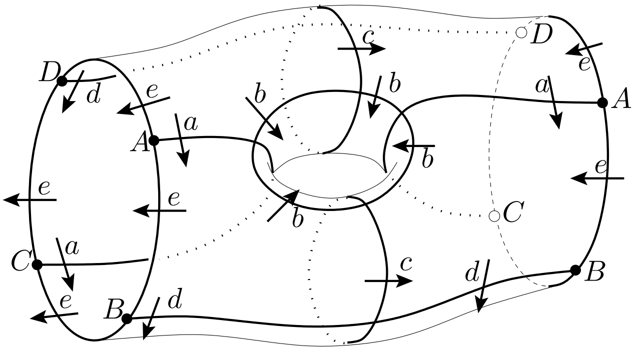
The fact that these embeddings are all quasi-isometric is a consequence of the method used in [4] to prove injectivity: namely, each embedding is realised as the homomorphism induced on the fundamental groups by a locally isometric embedding of CAT(0) cubical complexes. We refer the reader to the final sentence in the statement of Theorem 1 of [4].

4.2 Some HNN extensions.

Other hyperbolic groups can be obtained by taking HNN extensions of surfaces, and applying the Bestvina-Feighn combination theorem. These are two-dimensional but have boundary not homeomorphic to the circle. An explicit construction of such an example is given below. These examples necessarily have local cut points in the boundary. It is known from work of M. Kapovich and B. Kleiner [8] that the boundary of a one-ended Gromov hyperbolic group which is 1-dimensional and has no local cut points is homeomorphic to either the Sierpinski carpet or the Menger curve. (see [9], section 8, for a further discussion). This raises the following question which we are so far unable to resolve:

Question 15 Do the groups $Diff(D^2, \partial D^2, vol)$, PB_m, or $Mod(S)$, for S a closed orientable surface, admit quasi-isometric embeddings of 2-dimensional hyperbolic groups with boundary homeomorphic to either a Sierpinski carpet or a Menger curve?

Construction of HNN extension examples. Consider the genus 2 orientable surface F with dissection obtained as shown in Figure 5. This defines a quasi-isometric embedding of the group $\pi_1(F)$ in the right-angled Artin group $G(C_5)$ where C_5 denotes the five cycle graph. Now form a double cover F_2 of F by cutting along the e-curve and gluing two copies of the subsurface shown in Figure 5. Further modify the dissection by doubling each of the two e-curves (replacing it with two parallel curves labelled in the same way).



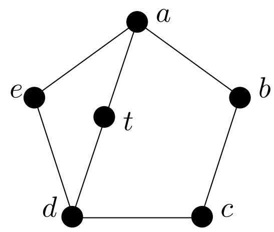


Figure 5: We obtain the surface F by identifying the two boundary circles of the surface on the left. The fundamental group of the complex X embeds in $G(\Delta)$, where Δ is the graph on the right.

Now between each pair of parallel e-curves we can define a simple closed path, labelled γ and γ', respectively. We construct a complex X by attaching an annulus A to our surface, joining the two curves γ and γ'. We extend the dissecting curves over the new annulus A, by drawing four segments which connect the two boundary components of A, and which are transversely labelled a and d in the obvious way. We also introduce a new dissecting curve, namely the core curve of the annulus, which shall be transverely labelled t.

We also modify the right-angled Artin group by adding a generator t to G which commutes with a and d, but not with e, b, or c. Thus $G = G(\Delta)$ for a new graph Δ containing C_5. It is easy to see that $G(\Delta)$ is of planar type (in fact Δ^{op} is a planar graph).

Now, the dissection of the surface, extended to the complex X, defines a homomorphism $\pi_1(X) \to G(\Delta)$. Moreover, by the technique of [4] this can easily be seen to be a quasi-isometric monomorphism: the complex X admits a CAT(0) squaring X_Q dual to the dissection, and the obvious labelling on the edges of this squaring determines a locally isometric embedding of X_Q into the standard cubical complex associated to $G(\Delta)$ (see [4] for more details).

Finally we observe that $\pi_1(X)$ is an HNN-extension. Choosing a basepoint in F_Q^2 and paths in F_Q^2 out to the endpoints of a t-edge E of the annulus A we define elements g, g' and t in $\pi_1(X)$ corresponding to the loops γ, γ' and the path E, respectively. We then have

$$\pi_1(X) = \pi_1(F_Q^2) \star_{t\colon \langle g\rangle \to \langle g'\rangle,\, t(g)=g'} = \pi_1(F_Q^2) \star \langle t\rangle / (tgt^{-1} = g')\,.$$

It follows from the Bestvina-Feighn Combination Theorem [3] that $\pi_1(X)$ is a word hyperbolic group. (Note that since the curves γ and γ' are non-parallel geodesics in F_Q^2 the hypotheses of [3] Corollary 2.3 are readily satisfied – namely, no powers of g and g' are conjugate in $\pi_1(F_Q^2)$, and g, g' are not proper powers of other elements).

4.3 The commutator subgroup of a right-angled Coxeter group.

In this subsection we present a natural quasi-isometric embedding of the commutator subgroup of a right-angled Coxeter group into the corresponding Artin group. We note that many Coxeter groups are Gromov hyperbolic groups. In particular, Januszkiewicz and Świątkowski [7] show that there exist Gromov hyperbolic right-angled Coxeter groups of virtual cohomological dimension n, for all $n \geqslant 1$.

Fix an arbitrary simplicial graph Δ and let $G(\Delta)$ be the associated right-angled Artin group, with standard generators $a_1, \ldots, a_n$. One may define the corresponding right-angled Coxeter group by adding the further relations that each generator has order 2:

$$W(\Delta) = G(\Delta)/\langle a_i^2 : i = 1, \ldots, n\rangle\,.$$

In the following we shall simply write $W = W(\Delta)$ and $G = G(\Delta)$. Observe that W is a group with 2-torsion. However, its commutator subgroup $[W, W]$ has index 2^n and is torsion free (as a consequence, for instance, of the following Lemma). An element $w \in W$ lies in $[W, W]$ precisely when each letter a_i appears an even number of times in any word representing w.

Lemma 16 *The group $[W, W]$ embeds quasi-isometrically in G.*

Proof We define a map $\phi\colon [W,W] \to G$ as follows. Let $w = b_1 b_2 \dots b_r$ be any word in the usual generators for an element $w \in [W,W]$. Thus $b_i \in \{a_1, \dots, a_n\}$. For each $i \in \{1, \dots, r\}$ we define $\epsilon_i \in \{\pm 1\}$ by $\epsilon_i = (-1)^{d+1}$ if b_i is the dth occurrence of that particular letter in the word w. Then we set $\phi(w) = b_1^{\epsilon_1} b_2^{\epsilon_2} \dots b_r^{\epsilon_r} \in G$. This gives a well-defined function since the element $\phi(w) \in G$ is invariant under modification of w by trivial insertion or deletion of a square a_i^2 and substitutions $a_i a_j \leftrightarrow a_j a_i$ for i, j adjacent in Δ. Moreover, since each letter a_j appears an even number of times in any word for an element $w \in [W,W]$, the above map defines a *homomorphism* $\phi\colon [W,W] \to G$. It is clear, since minimal length (or reduced) words for elements in $[W,W]$ are mapped to reduced words for elements in G, that ϕ is an injective homomorphism and a quasi-isometric embedding. $\square$

4.4 A closed hyperbolic 3-manifold group.

Let Υ denote a regular dodecahedron in $\mathbb{H}^3$ with dihedral angles $\pi/2$, and define W_Υ to be the right-angled Coxeter group generated by reflections in the faces of Υ. The defining graph for W_Υ as a right-angled Coxeter group shall be denoted Δ_Υ, and is isomorphic to the 1-skeleton of the icosahedron. We observe that $\Gamma_\Upsilon := [W_\Upsilon, W_\Upsilon]$ is the fundamental group of a closed oriented hyperbolic 3-manifold.

Corollary 17 *The closed hyperbolic 3-manifold group $\Gamma_\Upsilon = [W_\Upsilon, W_\Upsilon]$ may be embedded quasi-isometrically in a right-angled Artin group, namely in $G(\Delta_\Upsilon)$.*

Moreover, Figure 6 shows that the defining graph Δ_Υ is in fact of planar type. (This would be far from obvious at first sight. In fact, while the icosahedral graph Δ_Υ is planar, its complementary graph is not – it is obtained from Δ_Υ by adding an extra edge joining each antipodal pair of vertices, and this graph contains a $K_{3,3}$ graph embedded as a subgraph). Observe that in Figure 6 there are 12 circles, and each is disjoint from exactly 5 others whose incidence graph is in each case a 5-cycle. Thus we have here a circle diagram for a regular graph on 12 vertices with vertex valence 5, in which the link of every vertex spans a 5-cycle (the 5-cycle being self-dual). In other words, we have a circle diagram for the icosahedral graph Δ_Υ.

Thus the closed hyperbolic 3-manifold group Γ_Υ embeds quasi-isometrically in a pure braid group and in the group *Diff*$(D^2, \partial D^2, \mathit{vol})$.

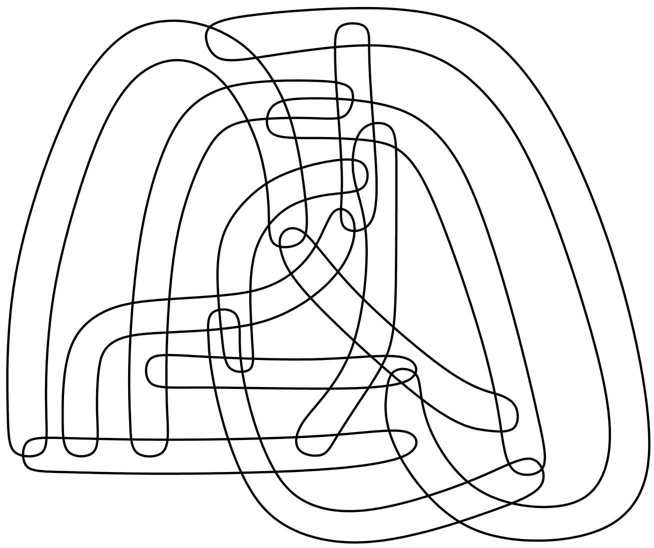


Figure 6: A planar circle diagram for the icosahedral graph Δ_Υ.

By Theorem 2 of [7], a Gromov hyperbolic right-angled Coxeter group can be a virtual n-manifold group only if $n \leqslant 4$. An example in dimension 4, similar to the 3-manifold group discussed above, would be provided by considering the group of reflections in the faces of a hyperbolic hyperdodecahedron (120-cell) with dihedral angles $\frac{\pi}{2}$. The commutator subgroup of this reflection group is a torsion free subgroup of index 2^{120}, and the fundamental group of a closed hyperbolic 4-manifold. While this group embeds in a closed surface mapping class group (by Corollary 5(i) and Lemma 16), we do not know whether it can be embedded in a braid group. To obtain such an embedding, it would be sufficient to find a planar circle diagram for the corresponding right-angled Artin group on 120 generators.

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